My research is devoted to *Analysis* and *Probability.* A significant part of my studies is concerned with *Gaussian stationary processes.* These are among the most widely applied models in statistics and engineering, while also commonly used within probability and theoretical physics. Nonetheless, many basic questions about them remain open. The interface between Probability and Analysis includes many other topics, among which I am interested in *concentration of measure*, *discrepancy theory* and *anti-concentration phenomena.* A summary of my work and future plans follows.

**Gassian Stationary Processes.**

Let $D \in \{\mathbb{R}, \mathbb{C}\}$ and $R \in \{\mathbb{R}, \mathbb{C}\}$. A Gaussian stationary process (GSP) is a random $f : D \to R$, whose finite marginal distributions are mean-zero multi-normal, and whose distribution is invariant to real shifts. Such a process is determined by its *covariance kernel* $\mathrm{cov}\,[f(t), f(s)] = r(t - \bar{s})$ (notice that the kernel is a function of one variable due to stationarity). The *spectral measure* of the process is the unique finite, non-negative measure on $\mathbb{R}$ satisfying $r(t) = \widehat{\rho}(t) = \int_{\mathbb{R}} e^{-i\lambda t}\, d\rho(\lambda)$.

**Horizontal density of zeroes.** Wiener [17, Ch. X] considered the model of a Stationary Gaussian Analytic function (stationary GAF), on a strip $D_\Delta = \{|\mathrm{Im}\, z| < \Delta\}$ with $0 < \Delta \le \infty$. This model includes some famous examples: $\sum_{n=0}^\infty w_n(\zeta_{2n}\cos(\lambda_n z) + \zeta_{2n+1}\sin(\lambda_n z))$, $\sum_{n\in\mathbb{Z}} \zeta_n \frac{\sin(\pi(z-n))}{z-n}$, $e^{-z^2/2}\sum_{n=0}^\infty \zeta_n \frac{z^n}{\sqrt{n!}}$, where $\zeta_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$ are i.i.d., and $w_n, \lambda_n \in \mathbb{R}$ are such that $\sum_n w_n^2 e^{\lambda_n y} < \infty$ for all $|y| < \Delta$.Wiener proved that, under some assumptions on the spectral measure, the zeroes of a stationary GAF in a strip obey the law of large numbers. He also computed a formula for their horizontal density. In [9] we generalize this result by removing those assumptions and by showing that the limit to be deterministic if and only if the spectral measure contains no atoms.
We also generalize these results to a natural counterpart of GAFs called *symmetric GAFs*, that is, Gaussian analytic functions $f : D \to \mathbb{C}$ on a domain $D \subset \mathbb{C}$ which posses a symmetry around the real axis: a.s., $\forall z \in D: \quad \overline{f(z)} = f(\bar{z})$. Examples are given by taking real coefficients $\zeta_n \sim \mathcal{N}_{\mathbb{R}}(0,1)$. In the course of this work, we develop a formula for the expected number of zeroes of any symmetric GAF in any domain. This generalizes works by Kac [15] and Shepp-Vanderbei [18].

It remains open to understand the horizontal density of zeroes for a GAF whose spectral measure contains atoms. In this case the limiting horizontal density of zeroes exists, but is non-deterministic. Another direction is investigating the *universality* of the above results, that is, extending them to non-Gaussian weakly stationary processes (e.g., when $\zeta_n$ are i.i.d. non-Gaussian).

**Fluctuations and limit theorems.** As before, let $f$ be a stationary GAF in the strip $D_\Delta$. In [10], we show that the variance of the number of zeroes of $f$ in $[0, T] \times [a, b]$ is asymptotically between $cT$ and $CT^2$ with positive constants $c$ and $C$. We also give conditions (in terms of the spectral measure) for the asymptotics to be exactly linear or quadratic in $T$. Recently with J. Buckley [4] we proved similar results for the winding number of a GSP $f : \mathbb{R} \to \mathbb{C}$ (which is a natural analogue of the number of zeros). We also prove that if $r, r'' \in L^2(\mathbb{R})$ then the winding obeys a central limit theorem. This establishes and extends works by physicists [8] who were motivated by modeling entanglement of polymers and flux lines of magnetic fields.

Fluctuations and CLT for the number of zeors of real GSPs were studied long ago in [6, 20]. However, their methods do not yield a closed formula for the variance (as our spectral methods do), and in particular it is not known whether the variance is always at least linear. Proving a CLT for complex zeroes with linear variance is still open. In addition, it is not clear what limit law, if at all, the zeroes (or winding) obey in case of super-linear variance. I believe that answering these questions will require unifying old and new tools, yielding a method which may be applied in other settings.

**Persistence.** For a GSP $f : \mathbb{R} \to \mathbb{R}$ we define the *persistence probability* $P_f(N)$ as the probability that $f > 0$ on the interval $[0, N]$. The question of estimating the asymptotic behavior of this probability as $N \to \infty$ is motivated by applications in engineering and physics, and was studied by many authors – including Rice, Slepian [19], and recently by Dembo-Mukherjee [7] and Krishna-Krishnapur [16]. In a recent work with O. Feldheim and S. Nitzan we were able to give, for the first time, a very general description of the qualitative behavior of $P_f(N)$. This description is given in terms of the behavior of the spectral measure near zero, and thus can be applied to processes whose covariance function is non-summable or changes sign. We also prove that if the spectral measure has a gap near the origin, then $\log P_f(N) \le -CN^2$, and if in addition it has a polynomial tail then $\log P_f(N) \le -e^{CN}$.

These results open the door to many interesting questions. In one dimension, the existence of persistence exponent, and a better understanding of the phenomena of *tiny persistence* is still lacking. Extensions to *higher dimensions* and to events of avoiding hitting *various domains* seem within reach with the new methods. Generalizations to *non-stationary* processes would be important for applications, though much more challenging. One such setting is the *membrane model*, a Gaussian process on a bounded domain of $\mathbb{Z}^d$ which is used to model an elastic membrane. Since in the radial direction the process resembles a stationary one, there is hope our methods may be applied.

**Concentration of zeroes.** Another interesting question is concerned with large deviations for the number of zeroes: what is the probability that the number of zeroes in a long interval $[0, T]$ differs from its mean by more than $\alpha T$ (for a given $\alpha > 0$)? Does this probability decay exponentially in $T$? Such a decay is known for certain complex zeros and nodal lines of random spherical harmonics. In [2] with R. Basu, A. Dembo, and O. Zeitouni we prove this phenomenon for zeros of smooth GSPs with summable correlations, using a combination of real and complex methods. It remains open to remove the summability condition, which we believe is unnecessary.

**Other Topics.**

**Concentration of measure.** The phenomena of *concentration of measure* may be roughly described as follows: if $\mu$ is a "nice" probability measure in $\mathbb{R}^n$, then any set of measure $\frac{1}{2}$, when enlarged by a small Euclidean ball will have almost a full $\mu$ measure. A common approach to reach concentration is via functional inequalities. Bobkov and Götze [5] studied the Poincaré property restricted to convex functions, which yields exponential concentration for *convex* sets. Together with Marsiglietti, Nayar and Wang [14], we give a full characterization of measures satisfying the convex Poincaré inequality on $\mathbb{R}$. Our result generalizes to product measures in $\mathbb{R}^n$, but generalization to other measures is open.

**Discrepancy.** Discrepancy theory is the study of well-distributed sets. One key conjecture in this area is the "small ball inequality", which got its name from an application to probability. The inequality is known to hold in two dimensions, but open in all others. Together with D. Bilyk [3] we give a new short proof in two dimensions, inspired by lacunary Fourier series, which yields a novel connection with binary nets. We hope that these insights may shed light on higher dimensions.

**Mean and minimum.** Together with O. Feldheim in [12], we show that for *any* independent $X, Y$ on $(0, \infty)$, it holds that $\liminf_{m\to\infty} \frac{\mathbb{P}(\min(X,Y)>m)}{\mathbb{P}(X+Y>2m)} = 0$. Moreover, our result may be used to study a model for *evolving social groups* introduced in [1]. In a future work, we plan to combined our results with additional random walk techniques in order to prove convergence of this model.

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